

Cosmological implications of Károlyházy uncertainty relation

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Károlyházy uncertainty relation, which can be viewed also as a relation between UV and IR scales in the framework of an effective quantum field theory satisfying a black hole entropy bound, strongly favors the existence of dark energy with its observed value. Here we estimate the dynamics of dark energy predicted by the Károlyházy relation during the cosmological evolution of the universe.

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I. INTRODUCTION

From the inception of quantum mechanics the concept of measurement (real or **Gedanken**) has proved to be a fundamental notion for revealing a genuine nature of physical reality. It is not without interest to address from this standpoint a notion of background space-time. That is, to ask how does the background space-time manifest itself in a way accessible to us in light of quantum mechanics and general relativity. General relativity treats space-time as a four-dimensional differentiable manifold with well defined metric structure of Minkowskian signature. Physically it is easy to understand that the introduction of a measuring device which undergoes quantum fluctuations does not allow one to measure a background space-time with unlimited accuracy. (In what follows we assume $\hbar = c = 1$). Namely, a measuring device (or simply a test body) with zero mean velocity, having a mass m and located within the region δx is characterized by the gravitating energy

$$E = m + \frac{\delta p^2}{m}, \quad (1)$$

where $\delta p \simeq \delta x^{-1}$. The measurement of a local characteristic of a background metric does not allow one to take δx very large and therefore after minimization of Eq.(1) with respect to mass, m , one gets an unavoidable disturbance of the background-space time. In 1959 Alden Mead showed (through a number of **Gedankenexperiments**) that the combination of Heisenberg uncertainty relations and general relativity puts absolute limitations on the sharpness of space-time structure at the Planck length $l_P \sim 10^{-33}\text{cm}$ [1]. It is instructive to quote briefly his discussion regarding the status of a fundamental length as this conceptual standpoint was unanimous in almost all subsequent papers about space-time uncertainties albeit many of the authors apparently did not know this paper: *"The idea is roughly as follows: Suppose there exists a fundamental length l . Since a space-time coordinate system, to be physically meaningful, must be referred to physical bodies, it follows that no Lorentzian coordinate system can be set up capable of specifying the coordinates*

of a space-time event more precisely than $\Delta x \gtrsim l$. Conversely, if the limitation on the coordinate system holds, the limitation on the localizability of particles follows immediately. Thus the fundamental length postulate may be equivalently stated as a postulate of a limitation on realizable coordinate system. . . . In terms of the light signal experiments this means, for instance, that the time required for a light signal to propagate from body A to body B and back (as measured by a clock at A) is subject to uncontrollable fluctuations. However, from the point of view of general relativity, it is completely equivalent to define the coordinates associated with each body and clock reading by some arbitrary convention, and to regard the light signal experiments as yielding information about the space-time metric associated with the coordinate system so defined. From this point of view, fluctuations in the results of light-signal experiments are to be regarded as indicating fluctuations in the metric, i.e., in the gravitational field. Thus it seems qualitatively plausible that a fundamental length postulate is equivalent to a postulate about gravitational field fluctuations." [1]. Mead's consideration tells us that any length undergoes fluctuations at least of the order of $\sim l_P$. Following this reasoning of discussion another interesting observation concerning a distance measurement for Minkowskian space-time was made by Károlyházy and his collaborators [2]. With respect to the Károlyházy uncertainty relation the distance t in Minkowski space-time can not be known to a better accuracy than

$$\delta t = \beta t_P^{2/3} t^{1/3}, \quad (2)$$

where β is a numerical factor of order unity. It is worth to notice that the papers [1, 2] sank into oblivion for a long time¹.

Following the discussion presented in paper [3] one can look at Eq.(2) as a relation between UV and IR scales in the framework of an effective quantum field theory satisfying black hole entropy bound. For an effective quantum

¹ As it is enlightened in [4], except for interest on the part of a few theorists who found the discussion of [1] convincing enough and felt that the idea of l_P as a possible fundamental length can be taken seriously, this idea seemed to be totally unacceptable for most of physicists. Many of the results of [1, 2] were "rediscovered" in 1980s and 1990s, see for instance [5].

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field theory in a box of size l with UV cutoff Λ the entropy S scales as, $S \sim l^3 \Lambda^3$. Nevertheless, considerations involving black holes demonstrate that the maximum entropy in a box of volume l^3 grows only as the area of the box. A consistent physical picture can be constructed by imposing a relationship between UV and infrared (IR) cutoffs [3]

$$l^3 \Lambda^3 \lesssim S_{BH} \simeq \left(\frac{l}{l_P} \right)^2, \quad (3)$$

where S_{BH} is the entropy of a black hole of size l . Consequently one arrives at the conclusion that the length l , which serves as an IR cutoff, cannot be chosen independently of the UV cutoff, and scales as Λ^{-3} . Rewriting this relation wholly in length terms, $\delta l \equiv \Lambda^{-1}$, one arrives at the Eq.(2).

II. MINKOWSKIAN SPACE-TIME IN LIGHT OF KÁROLYHÁZY UNCERTAINTY RELATION

Fluctuations of the Minkowski metric described by the Eq.(2) are characterized with a classical energy density (denoted hereafter by $\rho_{classical}$) [6]

$$\rho_{classical} \sim \frac{1}{t_P^{2/3} t^{10/3}}. \quad (4)$$

The relation (2) together with the time-energy uncertainty relation enables one to estimate a quantum energy density of the metric fluctuations of Minkowski space [6]. With respect to the Eq.(2) a length scale t can be known with a maximum precision δt determining thereby a minimal detectable cell $\delta t^3 \simeq t_P^2 t$ over a spatial region t^3 . In terms of the UV and IR scales discussed above one can look at the microstructure of space-time over a length scale t as consisting with cells $\delta t^3 \simeq t_P^2 t$. Such a cell represents a minimal detectable unit of space-time over a given length scale and if it has a finite age t , its existence due to time energy uncertainty relation can not be justified with energy smaller than $\sim t^{-1}$. Hence, having the above relation, Eq.(2), one concludes that if the age of the Minkowski space-time is t then over a spatial region with linear size t (determining the maximal observable patch) there exists a minimal cell δt^3 the energy of which due to time-energy uncertainty relation can not be smaller than

$$E_{\delta t^3} \gtrsim t^{-1}. \quad (5)$$

Hence, for energy density of metric fluctuations of Minkowski space one finds

$$\rho_{quantum} \sim \frac{E_{\delta t^3}}{\delta t^3} \sim \frac{1}{t_P^2 t^2}. \quad (6)$$

One can say the existence of this background energy density assures maximal stability of Minkowski space-time

against the fluctuations as the Eq.(2) determines maximal accuracy allowed by the nature. Similar ideas were elaborated in [7]. On the basis of the above arguments one can go further and see that due to Károlyházy relation, the energy E coming from the time energy uncertainty relation $E t \sim 1$ is determined with the accuracy $\delta E \sim E \delta t / t$. Respectively, one finds that the energy density $\rho = E / \delta t^3$ is characterized by the fluctuations $\delta \rho = \delta E / \delta t^3$ giving

$$\frac{\delta \rho}{\rho} \sim \frac{\delta t}{t} \sim \left(\frac{t_P}{t} \right)^{2/3}. \quad (7)$$

III. ENERGY BUDGET OF THE UNIVERSE DUE TO KÁROLYHÁZY RELATION

In the framework of inflationary cosmology the history of our universe encompasses inflationary stage followed by the radiation dominated and then matter dominated phases. The present cosmological data shows that we have already left the matter dominated phase for the dark energy dominated one which took place only recently (at $z \approx 0.3$) [8]. Generalization of our approach in presence of the energy components $\rho_{inflaton}$, $\rho_{radiation}$ and ρ_{matter} is straightforward. Due to time-energy uncertainty relation, the energy of the cell δt^3 determined by the space-time uncertainty relation can not be smaller than

$$E_{\delta t^3} \gtrsim t^{-1},$$

but now this energy contains a portion of the inflaton or radiation + matter energy that should be subtracted for estimating a quantum energy density of the metric fluctuations. In what follows we will assume a minimal time-energy uncertainty which seems reasonable as most of the history of the universe is successfully described by the thermal equilibrium approach. From this point of view, let us first consider the inflationary stage during of which scale factor grows nearly exponentially in time.

For simplicity let us take a pure de Sitter phase as an approximation to the inflationary stage. As it is shown in [6], the Eq.(2) is valid during the inflationary stage as well if the Hubble constant during inflation $H \lesssim m_P / 75^2 \approx 10^{-4} m_P$ (the timescale for the end of inflation is taken to be $\sim 75 H^{-1}$ [9]). During the inflation the energy of a cell δt^3 contains a fraction of inflaton energy of the order of $\sim \delta t^3 H^2 m_P^2$. Hence, the energy density of the background metric fluctuations takes the form

$$\rho_{quantum} \simeq \frac{1}{t \delta t^3} - \frac{3 H^2 m_P^2}{8 \pi}, \quad (8)$$

as long as

$$\frac{1}{t \delta t^3} - \frac{3 H^2 m_P^2}{8 \pi} \gtrsim \frac{1}{t_P^{2/3} t^{10/3}}, \quad (9)$$

i.e. $t \lesssim H^{-1}$ and then for $H^{-1} \lesssim t \lesssim 75H^{-1}$ follows its classical expression, Eq.(4), as there is no room left any more for the ρ_{quantum} due to inflaton energy. As the energy density of background metric fluctuations decays very fast during the inflation it does not affect appreciably the inflationary picture. To the end of inflation the fluctuations in the energy density implied by the Károlyházy uncertainty relation, Eq.(7), takes the form

$$\frac{\delta\rho}{\rho} \Big|_{t \sim 75H^{-1}} \sim \left(\frac{t_P H}{75} \right)^{2/3} \lesssim 10^{-5}, \text{ when } H \lesssim 10^{-6} m_P,$$

giving thereby a constraint on the Hubble constant during the inflation.

During the thermal history of the universe (that is, after the inflationary stage) a requirement due to time-energy uncertainty relation that the energy of the cell δt^3 can not be less than t^{-1} can be simply summarized by the relation

$$(\rho_{\text{radiation}} + \rho_{\text{matter}} + \rho_{\text{quantum}}) \delta t^3 \simeq t^{-1}. \quad (10)$$

So we get a sort of cosmic sum rule. This result immediately tells us that the value of ρ_{quantum} depends on the fractional contribution $\rho_{\text{radiation}} + \rho_{\text{matter}}$ to the Friedmann equation. Hence, when the energy density due to first two terms in Eq.(10) saturates the cosmic sum there remains almost no room for the dark energy ρ_{quantum} and it is given by its classical expression, Eq.(4), which is so small that can not be appreciable during the cosmological evolution. For Eq.(2) the relation (10) takes the form

$$(\rho_{\text{radiation}} + \rho_{\text{matter}} + \rho_{\text{quantum}}) \beta^3 t_P^2 t^2 \simeq 1, \quad (11)$$

which after relating t to the Hubble parameter looks similar to the cosmic sum rule obtained from Friedmann equation for a spatially flat metric. Namely, by taking into account that the age of the universe during its thermal history is $t = 1/2H$, $2/3H$ during radiation and matter dominated phases respectively², the relation (11) in light of the Friedmann equation

$$H^2 = \frac{8\pi t_P^2}{3} (\rho_{\text{radiation}} + \rho_{\text{matter}} + \rho_{\text{quantum}}), \quad (12)$$

tells us that not to create changes in the expansion rate of the universe at earlier epoches the parameter β should satisfy

$$\beta^3 \simeq \frac{32\pi}{3}, \quad \text{during the radiation domination,}$$

² For simplicity we assume an instantaneous transition from radiation domination to the matter domination.

and

$$\beta^3 \simeq \frac{72\pi}{12}, \quad \text{during the matter domination.}$$

By taking for the present (dark energy dominated) epoch $t \simeq H^{-1}$ from Eqs.(11, 12) one finds³

$$\beta^3 \simeq \frac{8\pi}{3}.$$

So, β should satisfy this relation if we want the universe to accelerate in the recent past, i.e., to get $\rho_{\text{quantum}} \gtrsim \rho_{\text{matter}}$ for the present epoch.

Certainly, a qualitative discussion based on the combination of uncertainty relations with the gravity for estimating an uncertainty in space-time distance measurement does not allow one to determine (in an unique manner) the parameter β in Eq.(2) with such a precision. But what seems interesting is that numerical estimates of β in various *Gedankenexperiments* [6, 11] are quite close to the above depicted values and, most important, as we have seen the predictive consistency of the Károlyházy uncertainty relation with the cosmology requires a slight decay of β during the cosmological evolution ($\beta \simeq 3.22, 2.66, 2.03$ during radiation, matter and dark energy dominated stages respectively), which may be attributed to the decay of radiation temperature of the universe, as it implies the decay of corresponding thermal fluctuations of the measuring device allowing one to perform a space-time measurement more precisely, or to the (almost inappreciable) short distance modification of gravity as the Károlyházy relation is based on the behavior of gravity at the distance $\sim \delta t$ (see for instance [6]), which for the time corresponding to the end of inflation, $t \sim 10^9 t_P$, to the radiation matter equality epoch, $t \sim 10^{32} t_P$, and to the present epoch, $t \sim 10^{60} t_P$, gives the length scales, $\sim 10^{-30} \text{cm}$, 10^{-22}cm , 10^{-13}cm , respectively⁴.

One can say the above described approach uses a minimal setup in a form of the basic principles of quantum mechanics and general relativity compared to the assumptions and conjectures underlying the basis for other approaches relating dark energy to the (micro)structure of space-time [13, 14, 15]. One of the key points used in these papers is to look at the cosmological constant as a canonically conjugate variable to the four volume of space-time and write down for those quantities uncertainty relation like to other Heisenberg relations. Another essential point used by these papers is to conjecture

³ The lower bound on the present age of the Universe can be established estimating the ages of various objects it consists of. For example, the temperature of the coldest white dwarfs in globular clusters yields a cluster age of $12.7 \pm 0.7 \text{Gyr}$ [10]. This gives $H_0 t_0 > 0.93 \pm 0.12$ in clear disagreement with the matter domination where the age is estimated as $2/3H_0$.

⁴ Even the scale $\sim 10^{-13} \text{cm}$ corresponding to the $t \sim 10^{60} t_P$ is much smaller than the present lower experimental bound on the Newtonian inverse square law [12].

the number of cells of space-time (which are usually considered to have the Planck size in contrast to what comes from the Károlyházy relation) to fluctuate according to the Poisson distribution. The most troublesome aspect of these approaches is that the ever-present Λ induced in such a way is hard to reconcile with the early cosmology [13, 14, 15, 16].

To summarize, let us start with the prescription concerning quantum calculus of space-time defined by the Károlyházy uncertainty relation [6]. The space-time uncertainty relation given by Eq.(2) is valid for Minkowskian space ([2], Ng and van Dam [5], [6]) as well as for de Sitter space during the inflationary stage [6] (that is, for space-time distances smaller or comparable to the duration of inflation $\sim 75H^{-1}$ [9]). The derivation of space-time uncertainty relation for some particular background space-time requires a separate consideration. Space-time uncertainty relation allows one to estimate the classical energy density of the corresponding metric fluctuations [6]. On the other hand it provides a way for estimating a quantum energy density of the metric fluctuations with the use of time-energy uncertainty relation. Namely, space-time uncertainty relation determines a minimal detectable cell δt^3 over a region with linear size t and if the space-time has a finite age, t , the energy of this cell can be estimated by using time-energy uncertainty relation $E_{\delta t^3} \simeq t^{-1}$ [6]. For estimating of energy density associated with the metric fluctuations during the cosmological evolution of the universe one should subtract the contribution of the inflaton or radiation+matter energy from the energy of δt^3 estimated through the time-energy uncertainty relation. In this way one gets a sort of cosmic sum rule, Eq.(10), which exhibits that by assuming a slight decay of β dur-

ing the cosmological evolution (*which is in the range of Gedankenexperiment estimates* [6], [11]) the Károlyházy uncertainty relation (2) can be simply reconciled with all cosmological epochs, that is, not to disturb appreciably the early cosmology and at the same time give a correct value for the dark energy density. On the other hand this slight decay of β can be understood either as a result of temperature decay of the radiation during the cosmological evolution which makes the measurement procedure more precise as it implies the decay of the corresponding thermal fluctuations of a measuring device or as a (almost inappreciable) short distance modification of gravity below the lengths scale $\sim 10^{-13}\text{cm}$. It should be emphasized that the relation (2) with a fixed value of β does not suffer from inconsistency as such with the cosmology if it satisfies

$$\beta^3 \simeq \frac{32\pi}{3},$$

but in this case it cannot provide sufficient amount of dark energy at present.

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